Mixed sheaves on Shimura varieties and their higher direct images in toroidal compactifications *

by

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0 Introduction

In this paper, we consider a *toroidal compactification* of a *mixed Shimura variety*

$$j : M \hookrightarrow M(\mathfrak{S}) \, .$$

According to [13], the *boundary* $M(\mathfrak{S}) - M$ has a natural *stratification* into locally closed subsets, each of which is itself (a quotient by the action of a finite group of) a Shimura variety. Let

$$i : M' \hookrightarrow M(\mathfrak{S})$$

be the inclusion of an individual such stratum. Both in the Hodge and the ℓ-adic context, there is a theory of *mixed sheaves*, and in particular, a functor

$$i^* j_*$$

from the bounded derived category of mixed sheaves on M to that of mixed sheaves on M'.

The objective of the present article is a formula for the effect of $i^* j_*$ on those complexes of mixed sheaves coming about via the *canonical construction*, denoted μ: The Shimura variety M is associated to a linear algebraic group P over $\mathbb{Q}$, and any complex of algebraic representations $\mathbb{V}^\bullet$ of P gives rise to a complex of mixed sheaves $\mu(\mathbb{V}^\bullet)$ on M. Let P' be the group belonging to M'; it is the quotient by a normal unipotent subgroup U' of a subgroup P_1 of P:

$$\begin{array}{ccccc} U' & \trianglelefteq & P_1 & \leq & P \\ & & \downarrow & & \\ & & P' & & \end{array}$$

Our main result (2.8 in the Hodge setting; 3.9 in the ℓ-adic setting) expresses the composition $i^* j_* \circ \mu$ in terms of the canonical construction μ' on M', and Hochschild cohomology of U'. It may be seen as complementing results of Harris and Zucker ([8]), and of Pink ([14]).

In the ℓ-adic setting, [14] treats the analogous question for the natural stratification of the *Baily–Borel compactification* M^* of a *pure* Shimura variety M. The resulting formula ([14] (5.3.1)) has a more complicated structure than ours: Besides Hochschild cohomology of a unipotent group, it also involves cohomology of a certain arithmetic group. Although we are interested in a different geometric situation, much of the abstract material developed in the first two sections of [14] will enter our proof. We should mention that the proof of Pink's result actually involves a toroidal compactification. The stratification used is the one induced by the stratification of M^*, and is therefore coarser than the one considered in the present work.

In [8], Harris and Zucker study the *Hodge structure* on the boundary cohomology of the *Borel–Serre compactification* of a Shimura variety. As in [14], toroidal compactifications enter the proof of the main result ([8] (5.5.2)).

It turns out to be necessary to control the structure of $i^*j_* \circ \mu(\mathbb{V}^\bullet)$ in the case when the stratum M' is minimal. There, the authors arrive at a description which is equivalent to ours ([8] (4.4.18)). Although they only treat the case of a pure Shimura variety, and do not relate their result directly to representations of the group P', it is fair to say that an important part of the main Hodge theoretic information entering our proof (see (b) below) is already contained in [8] (4.4). Still, our global strategy of proof of the main comparison result 2.8 is different: We employ Saito's *specialization functor*, and a homological yoga to reduce to two seemingly weaker comparison statements: (a) comparison for the full functor $i^*j_* \circ \mu$, but only on the level of local systems; (b) comparison on the level of variations of Hodge structure, but only for $\mathcal{H}^0 i^*j_* \circ \mu$.

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Notations and Conventions: Throughout the whole article, we make consistent use of the language and the main results of [13].

Algebraic representations of an algebraic group are finite dimensional by definition. If a group G acts on X, then we write $\mathrm{Cent}_G X$ for the kernel of the action. If Y is a subobject of X, then $\mathrm{Stab}_G Y$ denotes the subgroup of G stabilizing Y.

If X is a variety over $\mathbb{C}$, then $D^b_{\mathrm{c}}(X(\mathbb{C}))$ denotes the full triangulated subcategory of complexes of sheaves of abelian groups on $X(\mathbb{C})$ with constructible cohomology. The subcategory of complexes whose cohomology is *algebraically* constructible is denoted by $D^b_{\mathrm{c}}(X)$. If F is a coefficient field, then we define triangulated categories of complexes of sheaves of F-vector spaces

$$D^b_{\mathrm{c}}(X, F) \subset D^b_{\mathrm{c}}(X(\mathbb{C}), F)$$

in a similar fashion. The category $\mathbf{Perv}_F\, X$ is defined as the heart of the perverse t-structure on $D^b_{\mathrm{c}}(X, F)$.

Finally, the ring of finite adèles over $\mathbb{Q}$ is denoted by $\mathbb{A}_f$.

1 Strata in toroidal compactifications

This section provides complements to certain aspects of Pink's treatment ([13]). The first concerns the shape of the canonical stratification of a toroidal compactification of a Shimura variety. According to [13] 12.4 (c), these strata are quotients by finite group actions of "smaller" Shimura varieties. We shall show (1.6) that under mild restrictions (neatness of the compact group, and condition (+) below), the finite groups occurring are in fact trivial.

The second result concerns the formal completion of a stratum. Under the above restrictions, we show (1.13) that the completion in the toroidal compactification is canonically isomorphic to the completion in a suitable torus embedding. Under special assumptions on the cone decomposition giving rise to the compactification, this result is an immediate consequence of [13] 12.4 (c), which concerns the *closure* of the stratum in question.

Finally (1.17), we identify the normal cone of a stratum in a toroidal compactification.

Let $(P, \mathfrak{X})$ be *mixed Shimura data* ([13] Def. 2.1). So in particular, P is a connected algebraic linear group over $\mathbb{Q}$, and $P(\mathbb{R})$ acts on the complex manifold $\mathfrak{X}$ by analytic automorphisms. Any *admissible parabolic subgroup* ([13] Def. 4.5) Q of P has a canonical normal subgroup P_1 ([13] 4.7). There is a finite collection of *rational boundary components* $(P_1, \mathfrak{X}_1)$, indexed by the $P_1(\mathbb{R})$-orbits in $\pi_0(\mathfrak{X})$ ([13] 4.11). The $(P_1, \mathfrak{X}_1)$ are themselves mixed Shimura data.

Denote by W the unipotent radical of P. If P is reductive, i.e., if $W = 0$, then $(P, \mathfrak{X})$ is called *pure*.

Consider the following condition on $(P, \mathfrak{X})$:

(+) If G denotes the maximal reductive quotient of P, then the neutral connected component $Z(G)^0$ of the center $Z(G)$ of G is, up to isogeny, a direct product of a $\mathbb{Q}$-split torus with a torus T of compact type (i.e., $T(\mathbb{R})$ is compact) defined over $\mathbb{Q}$.

From the proof of [13] Cor. 4.10, one concludes:

Proposition 1.1 *If $(P, \mathfrak{X})$ satisfies (+), then so does any rational boundary component $(P_1, \mathfrak{X}_1)$.*

Denote by $U_1 \trianglelefteq P_1$ the "weight -2" part of P_1. It is abelian, normal in Q, and central in the unipotent radical W_1 of P_1.

Fix a connected component $\mathfrak{X}^0$ of $\mathfrak{X}$, and denote by $(P_1, \mathfrak{X}_1)$ the associated rational boundary component. There is a natural open embedding

$$\iota : \mathfrak{X}^0 \longrightarrow \mathfrak{X}_1$$

([13] 4.11, Prop. 4.15 (a)). If $\mathfrak{X}_1^0$ denotes the connected component of $\mathfrak{X}_1$ containing $\mathfrak{X}^0$, then the image of the embedding can be described by means of the map *imaginary part*

$$\mathrm{im} : \mathfrak{X}_1 \longrightarrow U_1(\mathbb{R})(-1) := \frac{1}{2\pi i} \cdot U_1(\mathbb{R}) \subset U_1(\mathbb{C})$$

of [13] 4.14: $\mathfrak{X}^0$ is the preimage of an open convex cone

$$C(\mathfrak{X}^0, P_1) \subset U_1(\mathbb{R})(-1)$$

under $\mathrm{im}\,|_{\mathfrak{X}_1^0}$ ([13] Prop. 4.15 (b)).

Let us indicate the definition of the map im: given $x_1 \in \mathfrak{X}_1^0$, there is exactly one element $u_1 \in U_1(\mathbb{R})(-1)$ such that $u_1^{-1}(x_1) \in \mathfrak{X}_1^0$ is real, i.e., the associated morphism of the Deligne torus

$$\mathrm{int}(u_1^{-1}) \circ h_{x_1} : \mathbb{S}_{\mathbb{C}} \longrightarrow P_{1,\mathbb{C}}$$

([13] 2.1) descends to $\mathbb{R}$. Define $\mathrm{im}(x_1) := u_1$.

We now describe the composition

$$\mathrm{im} \circ \iota : \mathfrak{X}^0 \longrightarrow U_1(\mathbb{R})(-1)$$

in terms of the group

$$H_0 := \{(z,\alpha) \in \mathbb{S} \times \mathrm{GL}_{2,\mathbb{R}} \mid N(z) = \det(\alpha)\}$$

of [13] 4.3. Let U_0 denote the copy of $\mathbb{G}_{a,\mathbb{R}}$ in H_0 consisting of elements

$$\left(1, \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}\right) .$$

According to [13] Prop. 4.6, any $x \in \mathfrak{X}$ defines a morphism

$$\omega_x : H_{0,\mathbb{C}} \longrightarrow P_{\mathbb{C}} .$$

Lemma 1.2 *Let $x \in \mathfrak{X}^0$. Then*

$$\mathrm{im}(\iota x) \in U_1(\mathbb{R})(-1)$$

lies in $\omega_x(U_0(\mathbb{R})(-1) - \{0\})$.

Proof. Since the associations

$$x \longmapsto \omega_x$$

and

$$x \longmapsto \mathrm{im}(\iota x)$$

are $(U(\mathbb{R})(-1))$-equivariant, we may assume that $\mathrm{im}(x) = 0$, i.e., that

$$h_x : \mathbb{S}_{\mathbb{C}} \longrightarrow P_{\mathbb{C}}$$

descends to $\mathbb{R}$. According to the proof of [13] Prop. 4.6,

$$\omega_x : H_{0,\mathbb{C}} \longrightarrow P_{\mathbb{C}}$$

then descends to $\mathbb{R}$. Now

$$h_{\iota x} : \mathbb{S}_{\mathbb{C}} \longrightarrow P_{1,\mathbb{C}} \hookrightarrow P_{\mathbb{C}}$$

is given by $\omega_x \circ h_\infty$, for a certain embedding

$$h_\infty : \mathbb{S}_{\mathbb{C}} \longrightarrow H_{0,\mathbb{C}}$$

([13] 4.3).

More concretely, as can be seen from [13] 4.2–4.3, there is a $\tau \in \mathbb{C} - \mathbb{R}$ such that on $\mathbb{C}$-valued points, we have

$$h_\infty : (z_1, z_2) \longrightarrow \left((z_1, z_2), \begin{pmatrix} z_1 z_2 & \tau(1 - z_1 z_2) \\ 0 & 1 \end{pmatrix} \right) .$$

Hence there is an element

$$u_0 \in U_0(\mathbb{R})(-1) - \{0\}$$

such that $\mathrm{int}(u_0^{-1}) \circ h_\infty$ descends to $\mathbb{R}$. But then $\omega_x(u_0)$ has the defining property of $\mathrm{im}(\iota x)$. **q.e.d.**

Let F be a field of characteristic 0. By definition of Shimura data, any algebraic representation

$$\mathbb{V} \in \mathbf{Rep}_F \, P$$

comes equipped with a natural weight filtration $W_\bullet$ (see [13] Prop. 1.4). Lemma 1.2 enables us to relate it to the weight filtration $M_\bullet$ of

$$\mathrm{Res}^P_{P_1}(\mathbb{V}) \in \mathbf{Rep}_F \, P_1 \ :$$

Proposition 1.3 *Let* $\mathbb{V} \in \mathbf{Rep}_F \, P$*, and* $T \in U_1(\mathbb{Q})$ *such that*

$$\pm \frac{1}{2\pi i} T \in C(\mathfrak{X}^0, P_1) .$$

Then the weight filtration of $\log T$ *relative to* $W_\bullet$ *([5] (1.6.13)) exists, and is identical to* $M_\bullet$*.*

Proof. Set $N := \log T$. Since $\mathrm{Lie}(U_1)$ is of weight -2, we clearly have

$$N M_i \subset M_{i-2} .$$

It remains to prove that

$$N^k : \mathrm{Gr}^M_{m+k} \mathrm{Gr}^W_m \mathbb{V} \longrightarrow \mathrm{Gr}^M_{m-k} \mathrm{Gr}^W_m \mathbb{V}$$

is an isomorphism. According to 1.2, there are $x \in \mathfrak{X}^0$ and $u_0 \in U_0(\mathbb{R})(-1) - \{0\}$ such that

$$\omega_x : H_{0,\mathbb{C}} \longrightarrow P_\mathbb{C}$$

maps u_0 to $\pm \frac{1}{2\pi i} T$. By definition, $M_\bullet$ is the weight filtration associated to the morphism

$$\omega_x \circ h_\infty : \mathbb{S}_\mathbb{C} \longrightarrow P_{1,\mathbb{C}} .$$

Our assertion has become one about representations of $H_{0,\mathbb{C}}$. But $\mathbf{Rep}_\mathbb{C} \, H_{0,\mathbb{C}}$ is semisimple, the irreducible objects being given by

$$\mathrm{Sym}^n V \otimes \chi ,$$

V the standard representation of $\mathrm{GL}_{2,\mathbb{C}}$, χ a character of $H_{0,\mathbb{C}}$ and $n \geq 1$. It is straightforward to show that for any such representation, the weight

filtration defined by h_∞ equals the monodromy weight filtration for $\log u_0$.

q.e.d.

Corollary 1.4 *Let $T \in U_1(\mathbb{Q})$ such that $\pm\frac{1}{2\pi i}T \in C(\mathfrak{X}^0, P_1)$. Then*

$$\mathrm{Cent}_W(T) = \mathrm{Cent}_W(U_1) = W \cap P_1 \; .$$

Proof. The inclusions "$\supset$" hold since the right hand side is contained in W_1, and U_1 is central in W_1. For the reverse inclusions, let us show that

$$\mathrm{Lie}\,(\mathrm{Cent}_W(T)) \subset \mathrm{Lie}\,W$$

is contained in the (weight ≤ -1)-part of the restriction of the adjoint representation

$$\mathrm{Lie}\,W \in \mathbf{Rep}_{\mathbb{Q}}\,P$$

to P_1. Observe that with respect to this representation, we have

$$\ker(\log T) = \mathrm{Lie}\,(\mathrm{Cent}_W(T)) \; .$$

First, recall ([13] 2.1) that $\mathrm{Gr}_m^{W_\bullet}(\mathrm{Lie}\,W) = 0$ for $m \ge 0$. From the defining property of the weight filtration $M_\bullet$ of $\log T$ relative to $W_\bullet$, it follows that

$$\ker(\log T) \subset M_{-1}\,(\mathrm{Lie}\,W) \; .$$

Proposition 1.3 guarantees that the right hand side equals the (weight ≤ -1)-part under the action of P_1. Our claim therefore follows from the equality

$$M_{-1}\,(\mathrm{Lie}\,W) = \mathrm{Lie}\,(W \cap P_1)$$

([13] proof of Lemma 4.8). **q.e.d.**

Lemma 1.5 *Let $P_1 \trianglelefteq Q \le P$ as before, let $\Gamma \le Q(\mathbb{Q})$ be contained in a compact subgroup of $Q(\mathbb{A}_f)$, and assume that Γ centralizes U_1. Then a subgroup of finite index in Γ is contained in*

$$(Z(P) \cdot P_1)(\mathbb{Q}) \; .$$

If (+) holds for $(P, \mathfrak{X})$, then a subgroup of finite index in Γ is contained in $P_1(\mathbb{Q})$.

Proof. The two statements are equivalent: if one passes from $(P, \mathfrak{X})$ to the quotient data $(P, \mathfrak{X})/Z(P)$ ([13] Prop. 2.9), then (+) holds. So assume that (+) is satisfied. Fix a point $x \in \mathfrak{X}$, and consider the associated homomorphism

$$\omega_x : H_{0,\mathbb{C}} \longrightarrow P_{\mathbb{C}} \; .$$

Since ω_x maps the subgroup U_0 of H_0 to U_1, the elements in the centralizer of U_1 also commute with $\omega_x(U_0)$.

First assume that $P = G = G^{\mathrm{ad}}$. By looking at the decomposition of $\operatorname{Lie} G_{\mathbb{R}}$ under the action of H_0 ([13] Lemma 4.4 (c)), one sees that the Lie algebra of the centralizer in $Q_{\mathbb{R}}$ of $\omega_x(U_0)$,

$$\operatorname{Lie}(\operatorname{Cent}_{Q_{\mathbb{R}}} U_0) \subset \operatorname{Lie} Q_{\mathbb{R}}$$

is contained in $\operatorname{Lie} P_{1,\mathbb{R}} + \operatorname{Lie}(\operatorname{Cent}_{G_{\mathbb{R}}} \operatorname{im}(\omega_x))$. But $\operatorname{Cent}_{G_{\mathbb{R}}} \operatorname{im}(\omega_x)$ is a compact group, hence the image of Γ in $(Q/P_1)(\mathbb{Q})$ is finite.

Next, if $P = G$, then by the above,

$$\Gamma \cap (Z(G) \cdot P_1)(\mathbb{Q})$$

is of finite index in Γ. Because of (+), the image of Γ in $(Q/P_1)(\mathbb{Q})$ is again finite.

In the general case,

$$\Gamma \cap (W \cdot P_1)(\mathbb{Q})$$

is of finite index in Γ. Analysing the decomposition of $\operatorname{Lie} W_{\mathbb{R}}$ under the action of H_0 ([13] Lemma 4.4 (a) and (b)), or using Corollary 1.4, one realizes that

$$\operatorname{Lie}(\operatorname{Cent}_Q U_1) \cap \operatorname{Lie} W \subset \operatorname{Lie} P_1 \ .$$

q.e.d.

The *Shimura varieties* associated to mixed Shimura data $(P, \mathfrak{X})$ are indexed by the open compact subgroups of $P(\mathbb{A}_f)$. If K is one such, then the analytic space of $\mathbb{C}$-valued points of the corresponding variety $M^K := M^K(P, \mathfrak{X})$ is given as

$$M^K(\mathbb{C}) := P(\mathbb{Q}) \backslash (\mathfrak{X} \times P(\mathbb{A}_f)/K) \ .$$

In order to discuss compactifications, we need to introduce the *conical complex* associated to $(P, \mathfrak{X})$: set-theoretically, it is defined as

$$\mathcal{C}(P, \mathfrak{X}) := \coprod_{(\mathfrak{X}^0, P_1)} C(\mathfrak{X}^0, P_1) \ .$$

By [13] 4.24, the conical complex is naturally equipped with a topology (which is usually different from the coproduct topology). The closure $C^*(\mathfrak{X}^0, P_1)$ of $C(\mathfrak{X}^0, P_1)$ inside $\mathcal{C}(P, \mathfrak{X})$ can still be considered as a convex cone in $U_1(\mathbb{R})(-1)$, with the induced topology.

For fixed K, the (partial) *toroidal compactifications* of M^K are parameterized by *K-admissible partial cone decompositions*, which are collections of subsets of

$$\mathcal{C}(P, \mathfrak{X}) \times P(\mathbb{A}_f)$$

satisfying the axioms of [13] 6.4. If $\mathfrak{S}$ is one such, then in particular any member of $\mathfrak{S}$ is of the shape

$$\sigma \times \{p\} \ ,$$

$p \in P(\mathbb{A}_f)$, $\sigma \subset C^*(\mathfrak{X}^0, P_1)$ a *convex rational polyhedral cone* in the vector space $U_1(\mathbb{R})(-1)$ ([13] 5.1) not containing any non-trivial linear subspace.

Let $M^K(\mathfrak{S}) := M^K(P, \mathfrak{X}, \mathfrak{S})$ be the associated compactification. It comes equipped with a natural stratification into locally closed strata, each of which looks as follows: Fix a pair $(\mathfrak{X}^0, P_1)$ as above, $p \in P(\mathbb{A}_f)$ and

$$\sigma \times \{p\} \in \mathfrak{S}$$

such that $\sigma \subset C^*(\mathfrak{X}^0, P_1)$. Assume that

$$\sigma \cap C(\mathfrak{X}^0, P_1) \neq \emptyset \; .$$

To σ, one associates Shimura data

$$\left(P_{1,[\sigma]}, \mathfrak{X}_{1,[\sigma]}\right)$$

([13] 7.1), whose underlying group $P_{1,[\sigma]}$ is the quotient of P_1 by the algebraic subgroup

$$\langle \sigma \rangle \subset U_1$$

satisfying $\mathbb{R} \cdot \sigma = \frac{1}{2\pi i} \cdot \langle \sigma \rangle(\mathbb{R})$. Set

$$K_1 := P_1(\mathbb{A}_f) \cap p \cdot K \cdot p^{-1} \; , \quad \pi_{[\sigma]} : P_1 \longrightarrow\!\!\!\!\!\rightarrow P_{1,[\sigma]} \; .$$

According to [13] 7.3, there is a canonical map

$$i(\mathbb{C}) : M^{\pi_{[\sigma]}(K_1)}(P_{1,[\sigma]}, \mathfrak{X}_{1,[\sigma]})(\mathbb{C}) \longrightarrow M^K(\mathfrak{S})(\mathbb{C}) := M^K(P, \mathfrak{X}, \mathfrak{S})(\mathbb{C})$$

whose image is locally closed. In fact, $i(\mathbb{C})$ is a quotient map onto its image.

Proposition 1.6 *Assume that $(P, \mathfrak{X})$ satisfies $(+)$, and that K is* neat *(see e.g. [13] 0.6). Then $i(\mathbb{C})$ is injective, i.e., it identifies $M^{\pi_{[\sigma]}(K_1)}(\mathbb{C})$ with a locally closed subspace of $M^K(\mathfrak{S})(\mathbb{C})$.*

Proof. Consider the group Δ_1 of [13] 6.18:

$$\begin{aligned} H_Q &:= \mathrm{Stab}_{Q(\mathbb{Q})}(\mathfrak{X}_1) \cap P_1(\mathbb{A}_f) \cdot p \cdot K \cdot p^{-1} \; , \\ \Delta_1 &:= H_Q / P_1(\mathbb{Q}) \; . \end{aligned}$$

The subgroup $\Delta_1 \le (Q/P_1)(\mathbb{Q})$ is arithmetic. According to [13] 7.3, the image under $i(\mathbb{C})$ is given by the quotient of $M^{\pi_{[\sigma]}(K_1)}(\mathbb{C})$ by a certain subgroup

$$\mathrm{Stab}_{\Delta_1}([\sigma]) = \mathrm{Stab}_{H_Q}([\sigma]) / P_1(\mathbb{Q}) \le \Delta_1 \; .$$

This stabilizer refers to the action of H_Q on the double quotient

$$P_1(\mathbb{Q}) \backslash \mathfrak{S}_1 / P_1(\mathbb{A}_f)$$

of [13] 7.3. Denote the projection $Q \to Q/P_1$ by pr, so $\Delta_1 = \mathrm{pr}(H_Q)$, and

$$\mathrm{Stab}_{\Delta_1}([\sigma]) = \mathrm{pr}\left(\mathrm{Stab}_{H_Q}([\sigma])\right) \; .$$

By Lemma 1.7, this group is trivial under the hypotheses of the proposition.

q.e.d.

Lemma 1.7 *If $(P, \mathfrak{X})$ satisfies $(+)$ then $\mathrm{Stab}_{\Delta_1}([\sigma])$ is finite. If, in addition, K is neat then $\mathrm{Stab}_{\Delta_1}([\sigma]) = 1$.*

Proof. The second claim follows from the first since $\mathrm{Stab}_{\Delta_1}([\sigma])$ is contained in

$$(Q/P_1)(\mathbb{Q}) \cap \mathrm{pr}(p \cdot K \cdot p^{-1}) \; ,$$

which is neat if K is.

Consider the arithmetic subgroup of $Q(\mathbb{Q})$

$$\Gamma_Q := H_Q \cap p \cdot K \cdot p^{-1} \; .$$

The group $\mathrm{pr}(\Gamma_Q)$ is arithmetic, hence of finite index in Δ_1. Hence

$$\mathrm{Stab}_{\mathrm{pr}(\Gamma_Q)}([\sigma]) = \mathrm{pr}\left(\mathrm{Stab}_{\Gamma_Q}([\sigma])\right) \le \mathrm{Stab}_{\Delta_1}([\sigma])$$

is of finite index. Now

$$\mathrm{Stab}_{\Gamma_Q}(\sigma) \le \mathrm{Stab}_{\Gamma_Q}([\sigma])$$

is of finite index. By [13] Thm. 6.19, a subgroup of finite index of $\mathrm{Stab}_{\Gamma_Q}(\sigma)$ centralizes U_1. Our claim thus follows from Lemma 1.5. **q.e.d.**

Remark 1.8 *The lemma shows that the groups "* $\mathrm{Stab}_{\Delta_1}([\sigma])$*" occurring in 7.11, 7.15, 7.17, 9.36, 9.37, and 12.4 of [13] are all trivial provided that $(P, \mathfrak{X})$ satisfies $(+)$ and K is neat.*

We continue the study of the map

$$i(\mathbb{C}) : M^{\pi_{[\sigma]}(K_1)}(\mathbb{C}) \longrightarrow M^K(\mathfrak{S})(\mathbb{C}) \; .$$

Let $\mathfrak{S}_{1,[\sigma]}$ be the minimal K_1-admissible cone decomposition of

$$\mathcal{C}(P_1, \mathfrak{X}_1) \times P_1(\mathbb{A}_f)$$

containing $\sigma \times \{1\}$; $\mathfrak{S}_{1,[\sigma]}$ can be realized inside the decomposition $\mathfrak{S}_1^0$ of [13] 6.13; by definition, it is *concentrated in the unipotent fibre* ([13] 6.5 (d)). View $M^{\pi_{[\sigma]}(K_1)}(\mathbb{C})$ as sitting inside $M^{K_1}(\mathfrak{S}_{1,[\sigma]})(\mathbb{C})$:

$$i_1(\mathbb{C}) : M^{\pi_{[\sigma]}(K_1)}(\mathbb{C}) \hookrightarrow M^{K_1}(\mathfrak{S}_{1,[\sigma]})(\mathbb{C}) \; .$$

Consider the diagram

$$\begin{tikzcd} M^{\pi_{[\sigma]}(K_1)}(\mathbb{C}) \arrow[r, hook, "i_1(\mathbb{C})"] \arrow[dr, hook, "i(\mathbb{C})"'] & M^{K_1}(\mathfrak{S}_{1,[\sigma]})(\mathbb{C}) \\ & M^K(\mathfrak{S})(\mathbb{C}) \end{tikzcd}$$

[13] 6.13 contains the definition of an open neighbourhood

$$\mathfrak{U} := \overline{\mathfrak{U}}(P_1, \mathfrak{X}_1, p)$$

of $M^{\pi_{[\sigma]}(K_1)}(\mathbb{C})$ in $M^{K_1}(\mathfrak{S}_{1,[\sigma]})(\mathbb{C})$, and a natural extension f of the map $i(\mathbb{C})$ to $\mathfrak{U}$:

$$\begin{tikzcd} M^{\pi_{[\sigma]}(K_1)}(\mathbb{C}) \arrow[r, hook] \arrow[dr, hook, "i(\mathbb{C})"'] & \mathfrak{U} \arrow[d, "f"] \\ & M^K(\mathfrak{S})(\mathbb{C}) \end{tikzcd}$$

Proposition 1.9 *(a) f is open.*
(b) We have the equality

$$f^{-1}(M^K(\mathbb{C})) = \mathfrak{U} \cap M^{K_1}(\mathbb{C}) \; .$$

Proof. Let us recall the definition of $\overline{\mathfrak{U}}(P_1, \mathfrak{X}_1, p)$, and part of the construction of $M^K(\mathfrak{S})(\mathbb{C})$: Let $\mathfrak{X}^+ \subset \mathfrak{X}$ be the preimage under

$$\mathfrak{X} \longrightarrow \pi_0(\mathfrak{X})$$

of the $P_1(\mathbb{R})$-orbit in $\pi_0(\mathfrak{X})$ associated to $\mathfrak{X}_1$, and

$$\mathfrak{X}^+ \longrightarrow \mathfrak{X}_1$$

the map discussed after Proposition 1.1; according to [13] Prop. 4.15 (a), it is still an open embedding (i.e., injective). As in [13] 6.10, set

$$\mathfrak{U}(P_1, \mathfrak{X}_1, p) := P_1(\mathbb{Q}) \backslash (\mathfrak{X}^+ \times P_1(\mathbb{A}_f)/K_1) \; .$$

It obviously admits an open embedding into $M^{K_1}(\mathbb{C})$ as well as an open morphism to $M^K(\mathbb{C})$. One defines ([13] 6.13)

$$\overline{\mathfrak{U}}(P_1, \mathfrak{X}_1, p) \subset M^{K_1}(\mathfrak{S}_{1,[\sigma]})(\mathbb{C})$$

as the interior of the closure of $\mathfrak{U}(P_1, \mathfrak{X}_1, p)$. Then $M^K(\mathfrak{S})(\mathbb{C})$ is defined as the quotient with respect to some equivalence relation $\sim$ on the disjoint sum of all $\overline{\mathfrak{U}}(P_1, \mathfrak{X}_1, p)$ ([13] 6.24). In particular, for our *fixed* choice of $(P_1, \mathfrak{X}_1)$ and p, there is a continuous map

$$f : \overline{\mathfrak{U}}(P_1, \mathfrak{X}_1, p) \longrightarrow M^K(\mathfrak{S})(\mathbb{C}) \; .$$

From the description of $\sim$ ([13] 6.15–6.16), one sees that f is open; the central point is that the maps

$$\overline{\beta} := \overline{\beta}(P_1, \mathfrak{X}_1, P_1', \mathfrak{X}_1', p) : \overline{\mathfrak{U}}(P_1, \mathfrak{X}_1, p) \cap M^{K_1}(P_1, \mathfrak{X}_1, \mathfrak{S}''^0)(\mathbb{C}) \longrightarrow \overline{\mathfrak{U}}(P_1', \mathfrak{X}_1', p)$$

of [13] page 152 are open. This shows (a). As for (b), one observes that

$$\overline{\beta}^{-1} (\mathfrak{U}(P_1', \mathfrak{X}_1', p)) = \mathfrak{U}(P_1, \mathfrak{X}_1, p) \; .$$

q.e.d.

Remark 1.10 *[13] Cor. 7.17 gives a much stronger statement than Proposition 1.9 (a), assuming that $\mathfrak{S}$ is* complete *([13] 6.4) and satisfies condition*

(∗) of [13] 7.12. In this case, one can identify a suitable open neighbourhood of the closure *of*

$$\mathrm{Stab}_{\Delta_1}([\sigma])\backslash M^{\pi_{[\sigma]}(K_1)}(\mathbb{C}) = \mathrm{Im}(i(\mathbb{C})) \subset M^K(\mathfrak{S})(\mathbb{C})$$

with an open neighbourhood of the closure of

$$\mathrm{Stab}_{\Delta_1}([\sigma])\backslash M^{\pi_{[\sigma]}(K_1)}(\mathbb{C}) \subset \mathrm{Stab}_{\Delta_1}([\sigma])\backslash M^{K_1}(\mathfrak{S}_1)(\mathbb{C}) \ ,$$

where

$$\mathfrak{S}_{1,[\sigma]} \subset \mathfrak{S}_1 := ([\cdot p]^* \mathfrak{S}) \mid_{(P_1, \mathfrak{X}_1)}$$

([13] 6.5 (a) and (c)).

Consequently, one can identify the formal completions (in the sense of analytic spaces) of $M^K(\mathfrak{S})(\mathbb{C})$ and of

$$\mathrm{Stab}_{\Delta_1}([\sigma])\backslash M^{K_1}(\mathfrak{S}_1)(\mathbb{C})$$

along the closure of the stratum

$$\mathrm{Stab}_{\Delta_1}([\sigma])\backslash M^{\pi_{[\sigma]}(K_1)}(\mathbb{C}) \ .$$

It will be important to know that without the hypotheses of [13] Cor. 7.17, the completions along the stratum itself still agree. For simplicity, we assume that the hypotheses of Proposition 1.6 are met, and hence that $\mathrm{Stab}_{\Delta_1}([\sigma]) = 1$.

Theorem 1.11 *Assume that $(P, \mathfrak{X})$ satisfies $(+)$, and that K is neat.*

(i) The map f of 1.9 is locally biholomorphic near $M^{\pi_{[\sigma]}(K_1)}(\mathbb{C})$.

(ii) f induces an isomorphism between the formal analytic completions of $M^K(\mathfrak{S})(\mathbb{C})$ and of $M^{K_1}(\mathfrak{S}_{1,[\sigma]})(\mathbb{C})$ along $M^{\pi_{[\sigma]}(K_1)}(\mathbb{C})$.

Proof. f is open and identifies the analytic subsets

$$M^{\pi_{[\sigma]}(K_1)}(\mathbb{C}) \subset M^{K_1}(\mathfrak{S}_{1,[\sigma]})(\mathbb{C})$$

and

$$M^{\pi_{[\sigma]}(K_1)}(\mathbb{C}) \subset M^K(\mathfrak{S})(\mathbb{C}) \ .$$

For (ii), we have to compare certain sheaves of functions. The claim therefore follows from (i).

According to [13] 6.18, the image of f equals the quotient of $\mathfrak{U}$ by the action of a group Δ_1 of analytic automorphisms, which according to [13] Prop. 6.20 is properly discontinuous. **q.e.d.**

So far, we have worked in the category of analytic spaces. According to Pink's generalization to mixed Shimura varieties of the Algebraization Theorem of Baily and Borel ([13] Prop. 9.24), there exist canonical structures of normal algebraic varieties on the $M^K(P, \mathfrak{X})(\mathbb{C})$, which we denote as

$$M^K_{\mathbb{C}} := M^K(P, \mathfrak{X})_{\mathbb{C}} \ .$$

If there exists a structure of normal algebraic variety on $M^K(P,\mathfrak{X},\mathfrak{S})(\mathbb{C})$ extending $M^K_{\mathbb{C}}$, then it is unique ([13] 9.25); denote it as

$$M^K(\mathfrak{S})_{\mathbb{C}} := M^K(P,\mathfrak{X},\mathfrak{S})_{\mathbb{C}} \; .$$

Pink gives criteria on the existence of $M^K(\mathfrak{S})_{\mathbb{C}}$ ([13] 9.27, 9.28). If any cone of a cone decomposition $\mathfrak{S}'$ for $(P,\mathfrak{X})$ is contained in a cone of $\mathfrak{S}$, and both $M^K(\mathfrak{S}')_{\mathbb{C}}$ and $M^K(\mathfrak{S})_{\mathbb{C}}$ exist, then the morphism

$$M^K(\mathfrak{S}')(\mathbb{C}) \longrightarrow M^K(\mathfrak{S})(\mathbb{C})$$

is algebraic ([13] 9.25). From now on we implicitly assume the existence whenever we talk about $M^K(\mathfrak{S})_{\mathbb{C}}$.

According to [13] Prop. 9.36, the stratification of $M^K(\mathfrak{S})_{\mathbb{C}}$ holds algebraically.

Theorem 1.12 *Assume that $(P,\mathfrak{X})$ satisfies $(+)$, and that K is neat. The isomorphism of Theorem 1.11 induces a canonical isomorphism between the formal completions of $M^K(\mathfrak{S})_{\mathbb{C}}$ and of $M^{K_1}(\mathfrak{S}_{1,[\sigma]})_{\mathbb{C}}$ along $M_{\mathbb{C}}^{\pi_{[\sigma]}(K_1)}$.*

Proof. If $\mathfrak{S}$ is complete and satisfies $(*)$ of [13] 7.12, then this is an immediate consequence of [13] Prop. 9.37, which concerns the formal completions along the closure of $M_{\mathbb{C}}^{\pi_{[\sigma]}(K_1)}$.

We may replace K by a normal subgroup K' of finite index: the objects on the level of K come about as quotients under the finite group K/K' of those on the level of K'. Therefore, we may assume, thanks to [13] Prop. 9.29 and Prop. 7.13, that there is a complete cone decomposition $\mathfrak{S}'$ containing $\sigma \times \{p\}$ and satisfying $(*)$ of [13] 7.12. Let $\mathfrak{S}''$ be the coarsest refinement of both $\mathfrak{S}$ and $\mathfrak{S}'$; it still contains $\sigma \times \{p\}$, and $M^K(\mathfrak{S}'')_{\mathbb{C}}$ exists because of [13] Prop. 9.28. We have

$$\mathfrak{S}_{1,[\sigma]} = \mathfrak{S}''_{1,[\sigma]} = \mathfrak{S}'_{1,[\sigma]} \; ,$$

hence the formal completions all agree analytically. But on the level of $\mathfrak{S}'_{1,[\sigma]}$, the isomorphism is algebraic. **q.e.d.**

According to [13] Thm. 11.18, there exists a *canonical model* of the variety $M^K(P,\mathfrak{X})_{\mathbb{C}}$, which we denote as

$$M^K := M^K(P,\mathfrak{X}) \; .$$

It is defined over the *reflex field* $E(P,\mathfrak{X})$ of $(P,\mathfrak{X})$ ([13] 11.1). The reflex field does not change when passing from $(P,\mathfrak{X})$ to a rational boundary component ([13] Prop. 12.1).

If $M^K(\mathfrak{S})_{\mathbb{C}}$ exists, then it has a canonical model $M^K(\mathfrak{S})$ over $E(P,\mathfrak{X})$ extending M^K, and the stratification descends to $E(P,\mathfrak{X})$. In fact, [13] Thm. 12.4 contains these statements under special hypotheses on $\mathfrak{S}$. However, one passes from $\mathfrak{S}$ to a covering by finite cone decompositions (corresponding to an open covering of $M^K(\mathfrak{S})_{\mathbb{C}}$), and then ([13] Cor. 9.33) to a subgroup of K of finite index to see that the above claims hold as soon as $M^K(\mathfrak{S})_{\mathbb{C}}$ exists.

Theorem 1.13 *Assume that* $(P, \mathfrak{X})$ *satisfies* $(+)$*, and that* K *is neat. The isomorphism of Theorem 1.12 descends to a canonical isomorphism between the formal completions of* $M^K(\mathfrak{S})$ *and of* $M^{K_1}(\mathfrak{S}_{1,[\sigma]})$ *along* $M^{\pi_{[\sigma]}(K_1)}$.

Proof. If $\mathfrak{S}$ is complete and satisfies $(*)$ of [13] 7.12, then this statement is contained in [13] Thm. 12.4 (c).

In fact, the proof of [13] Thm. 12.4 (c) does not directly use the special hypotheses on $\mathfrak{S}$: the strategy is really to prove 1.13 and then deduce the stronger conclusion of [13] 12.4 (c) from the fact that it holds over $\mathbb{C}$; the point there is ([13] 12.6) that since the schemes are normal, morphisms descend if they descend on some open dense subscheme.

Thus the proof of our claim is contained in [13] 12.7–12.17. **q.e.d.**

Remark 1.14 *(a) Without any hypotheses on* $(P, \mathfrak{X})$ *and* K*, there are obvious variants of Theorems 1.11, 1.12, and 1.13. In particular, there is a canonical isomorphism between the formal completions of* $M^K(\mathfrak{S})$ *and of*

$$\mathrm{Stab}_{\Delta_1}([\sigma]) \backslash M^{K_1}(\mathfrak{S}_{1,[\sigma]})$$

along

$$\mathrm{Stab}_{\Delta_1}([\sigma]) \backslash M^{\pi_{[\sigma]}(K_1)} \; .$$

(b) By choosing simultaneous refinements, one sees that the isomorphisms of 1.11 (ii), 1.12, and 1.13 do not depend on the cone decomposition $\mathfrak{S}$ *"surrounding" our fixed cone* $\sigma \times \{p\}$*.*

In the situation we have been considering, the cone σ is called *smooth* with respect to the lattice

$$\Gamma_U^p(-1) := \frac{1}{2\pi i} \cdot (U_1(\mathbb{Q}) \cap K_1) \subset \frac{1}{2\pi i} \cdot U_1(\mathbb{R})$$

if the semi-group

$$\Lambda_\sigma := \sigma \cap \Gamma_U^p(-1)$$

can be generated (as semi-group) by a subset of a $\mathbb{Z}$-basis of $\Gamma_U^p(-1)$. The corresponding statement is then necessarily true for any face of σ. Hence the K_1-admissible partial cone decomposition $\mathfrak{S}_{1,[\sigma]}$ is smooth in the sense of [13] 6.4.

Let us introduce the following condition on $(P_1, \mathfrak{X}_1)$:

($\cong$) The canonical morphism $(P_{1,[\sigma]}, \mathfrak{X}_{1,[\sigma]}) \longrightarrow (P_1, \mathfrak{X}_1)/\langle\sigma\rangle$ ([13] 7.1) is an isomorphism.

In particular, there is an epimorphism of Shimura data from $(P_1, \mathfrak{X}_1)$ to $(P_{1,[\sigma]}, \mathfrak{X}_{1,[\sigma]})$. According to [13] 7.1, we have:

Proposition 1.15 *Condition* ($\cong$) *is satisfied whenever* $(P_1, \mathfrak{X}_1)$ *is a* proper *boundary component of some other mixed Shimura data, e.g., if the parabolic subgroup* $Q \leq P$ *is proper.*

Under the hypothesis ($\cong$), we can establish more structural properties of our varieties:

Lemma 1.16 *Assume that* ($\cong$) *is satisfied.*

(i) The Shimura variety M^{K_1} *is a torus torsor over* $M^{\pi_{[\sigma]}(K_1)}$*:*

$$\pi_{[\sigma]} : M^{K_1} \longrightarrow M^{\pi_{[\sigma]}(K_1)} \ .$$

The compactification $M^{K_1}(\mathfrak{S}_{1,[\sigma]})$ *is a* torus embedding *along the fibres of* $\pi_{[\sigma]}$*:*

$$\overline{\pi_{[\sigma]}} : M^{K_1}(\mathfrak{S}_{1,[\sigma]}) \longrightarrow M^{\pi_{[\sigma]}(K_1)}$$

admitting only one closed stratum. The section

$$i_1 : M^{\pi_{[\sigma]}(K_1)} \hookrightarrow M^{K_1}(\mathfrak{S}_{1,[\sigma]})$$

of $\overline{\pi_{[\sigma]}}$ *identifies the base with this closed stratum.*

(ii) Assume that σ *is smooth. Then*

$$\overline{\pi_{[\sigma]}} : M^{K_1}(\mathfrak{S}_{1,[\sigma]}) \longrightarrow M^{\pi_{[\sigma]}(K_1)}$$

carries a canonical structure of vector bundle, with zero section i_1*. The rank of this vector bundle is equal to the dimension of* σ*.*

Proof. (i) This is [13] remark on the bottom of page 165, taking into account that $\mathfrak{S}_{1,[\sigma]}$ is minimal with respect to the property of containing σ.
(ii) If σ is smooth of dimension c, then by definition, the semi-group Λ_σ can be generated by an appropriate basis of the ambient real vector space. One shows that each choice of such a basis gives rise to the same $M^{\pi_{[\sigma]}(K_1)}$-linear structure on $M^{K_1}(\mathfrak{S}_{1,[\sigma]})$. **q.e.d.**

We conclude the section by putting together all the results obtained so far:

Theorem 1.17 *Assume that* $(P, \mathfrak{X})$ *satisfies* $(+)$*, that* $(P_1, \mathfrak{X}_1)$ *satisfies* ($\cong$)*, that* K *is neat, and that* σ *is smooth. Then there is a canonical isomorphism of vector bundles over* $M^{\pi_{[\sigma]}(K_1)}$

$$\iota_\sigma : M^{K_1}(\mathfrak{S}_{1,[\sigma]}) \stackrel{\sim}{\longrightarrow} N_{M^{\pi_{[\sigma]}(K_1)} / M^K(\mathfrak{S})}$$

identifying $M^{K_1}(\mathfrak{S}_{1,[\sigma]})$ *and the normal bundle of* $M^{\pi_{[\sigma]}(K_1)}$ *in* $M^K(\mathfrak{S})$*.*

Proof. The isomorphism of Theorem 1.13 induces an isomorphism

$$N_{M^{\pi_{[\sigma]}(K_1)} / M^{K_1}(\mathfrak{S}_{1,[\sigma]})} \stackrel{\sim}{\longrightarrow} N_{M^{\pi_{[\sigma]}(K_1)} / M^K(\mathfrak{S})} \ .$$

But the normal bundle of the zero section in a vector bundle is canonically isomorphic to the vector bundle itself. **q.e.d.**

2 Higher direct images for Hodge modules

Let $M^K(\mathfrak{S}) = M^K(P, \mathfrak{X}, \mathfrak{S})$ be a toroidal compactification of a Shimura variety $M^K = M^K(P, \mathfrak{X})$, and $M^{\pi_{[\sigma]}(K_1)} = M^{\pi_{[\sigma]}(K_1)}(P_{1,[\sigma]}, \mathfrak{X}_{1,[\sigma]})$ a boundary stratum. Consider the situation

$$M^K \overset{j}{\hookrightarrow} M^K(\mathfrak{S}) \overset{i}{\hookleftarrow} M^{\pi_{[\sigma]}(K_1)} \ .$$

Saito's formalism ([16]) gives a functor $i^* j_*$ between the bounded derived categories of *algebraic mixed Hodge modules* on $M^K_{\mathbb{C}}$ and on $M^{\pi_{[\sigma]}(K_1)}_{\mathbb{C}}$ respectively. The main result of this section (Theorem 2.8) gives a formula for the restriction of $i^* j_*$ onto the image of the natural functor associating to an algebraic representation of P a variation of Hodge structure on $M^K_{\mathbb{C}}$. The proof has two steps: first, one employs the *specialization functor* à la Verdier–Saito, and Theorem 1.17, to reduce from the toroidal to a toric situation (2.6). The second step consists in proving the compatibility statement on the level of $\mathcal{H}^0$ and then appealing to homological algebra, which implies compatibility on the level of functors between derived categories.

Throughout the whole section, we fix a set of data satisfying the hypotheses of Theorem 1.17. We thus have Shimura data $(P, \mathfrak{X})$ satisfying condition $(+)$, a rational boundary component $(P_1, \mathfrak{X}_1)$ satisfying condition $(\cong)$, an open, compact and neat subgroup $K \le P(\mathbb{A}_f)$, an element $p \in P(\mathbb{A}_f)$ and a smooth cone $\sigma \times \{p\} \subset C^*(\mathfrak{X}^0, P_1) \times \{p\}$ belonging to some K-admissible partial cone decomposition $\mathfrak{S}$ such that $M^K(\mathfrak{S})$ exists. We assume that

$$\sigma \cap C(\mathfrak{X}^0, P_1) \neq \emptyset \ ,$$

and write $K_1 := P_1(\mathbb{A}_f) \cap p \cdot K \cdot p^{-1}$,

$$j : M^K \hookrightarrow M^K(\mathfrak{S}) \ ,$$

and

$$i : M^{\pi_{[\sigma]}(K_1)} \hookrightarrow M^K(\mathfrak{S}) \ .$$

Similarly, write

$$j_1 : M^{K_1} \hookrightarrow M^{K_1}(\mathfrak{S}_{1,[\sigma]}) \ ,$$

and

$$i_1 : M^{\pi_{[\sigma]}(K_1)} \hookrightarrow M^{K_1}(\mathfrak{S}_{1,[\sigma]})$$

for the immersions into the torus embedding $M^{K_1}(\mathfrak{S}_{1,[\sigma]})$, which according to Theorem 1.17 we identify with the normal bundle of $M^{\pi_{[\sigma]}(K_1)}$ in $M^K(\mathfrak{S})$.

If we denote by c the dimension of σ, then both i and i_1 are of pure codimension c.

The immersions $i(\mathbb{C})$ and $i_1(\mathbb{C})$ factor as

$$\begin{array}{ccccccc} M^{\pi_{[\sigma]}(K_1)}(\mathbb{C}) & \hookrightarrow & \mathfrak{U} & \hookrightarrow & M^{K_1}(\mathfrak{S}_{1,[\sigma]})(\mathbb{C}) & \hookleftarrow & M^{K_1}(\mathbb{C}) \\ \| & & \downarrow{\scriptstyle f} & & & & \\ M^{\pi_{[\sigma]}(K_1)}(\mathbb{C}) & \hookrightarrow & \mathfrak{V} & \hookrightarrow & M^{K}(\mathfrak{S})(\mathbb{C}) & \hookleftarrow & M^{K}(\mathbb{C}) \end{array}$$

where $\mathfrak{U}$ and $\mathfrak{V}$ are open subsets of the respective compactifications, and f is the map of 1.9. For a sheaf $\mathcal{F}$ on $M^K(\mathbb{C})$, we can consider the restriction $f^{-1}\mathcal{F}$ on $f^{-1}(M^K(\mathbb{C})) = \mathfrak{U} \cap M^{K_1}(\mathbb{C})$.

Let F be a coefficient field of characteristic 0. Denote by

$$\mu_{K,\mathrm{top}} : \mathbf{Rep}_F\, P \longrightarrow \mathbf{Loc}_F\, M^K(\mathbb{C})$$

the exact tensor functor associating to an algebraic representation $\mathbb{V}$ the sheaf of sections of

$$P(\mathbb{Q})\backslash\left(\mathfrak{X} \times \mathbb{V} \times P(\mathbb{A}_f)/K\right)$$

on

$$M^K(\mathbb{C}) = P(\mathbb{Q})\backslash\left(\mathfrak{X} \times P(\mathbb{A}_f)/K\right) \ .$$

Proposition 2.1 *Let $\mathbb{V} \in \mathbf{Rep}_F\, P$. Then $f^{-1} \circ \mu_{K,\mathrm{top}}\mathbb{V}$ is the restriction to $f^{-1}(M^K(\mathbb{C}))$ of the local system $\mu_{K_1,\mathrm{top}} \operatorname{Res}^P_{P_1} \mathbb{V}$ on*

$$M^{K_1}(\mathbb{C}) = P_1(\mathbb{Q})\backslash(\mathfrak{X}_1 \times P_1(\mathbb{A}_f)/K_1) \ .$$

Proof. $f^{-1}(M^K(\mathbb{C}))$ equals the set

$$\mathfrak{U}(P_1, \mathfrak{X}_1, p) := P_1(\mathbb{Q})\backslash\left(\mathfrak{X}^+ \times P_1(\mathbb{A}_f)/K_1\right) \subset M^{K_1}(\mathbb{C}) \ ,$$

and $f\,|_{\mathfrak{U}(P_1,\mathfrak{X}_1,p)}$ is given by

$$[(x, p_1)] \longmapsto [(x, p_1 \cdot p)]$$

([13] 6.10). **q.e.d.**

Using Theorem 1.11 (i), it is easy to construct a canonical isomorphism of functors

$$i^* j_* \quad , \quad i_1^*(j_1)_* \circ f^{-1} : D^b_{\mathrm{c}}(M^K(\mathbb{C})) \longrightarrow D^b_{\mathrm{c}}(M^{\pi_{[\sigma]}(K_1)}(\mathbb{C})) \ .$$

For us, it will be necessary to establish a connection between j_* and $(j_1)_* \circ f^{-1}$. This relation will be given by Verdier's specialization functor ([18] 9)

$$Sp_\sigma := Sp_{M^{\pi_{[\sigma]}(K_1)}} : D^b_{\mathrm{c}}(M^K(\mathfrak{S})(\mathbb{C}), F) \longrightarrow D^b_{\mathrm{c}}(M^{K_1}(\mathfrak{S}_{1,[\sigma]})(\mathbb{C}), F) \ .$$

According to [18] p. 358, the functor Sp_σ has the properties (SP0)–(SP6) of [18] 8, *convenablement transposées.* In particular:

(SP0) It can be computed locally.

(SP1) The complexes in the image of Sp_σ are *monodromic*, i.e., their cohomology objects are locally constant on each $\mathbb{C}^*$-orbit in $M^{K_1}(\mathfrak{S}_{1,[\sigma]})(\mathbb{C})$.

(SP5) We have the equality $i^* = i_1^* \circ Sp_\sigma$.

From Theorem 1.11 (i) and from (SP0), we conclude that in order to compute the effect of Sp_σ on a complex of sheaves $\mathcal{F}^\bullet$, we may pass to the complex $f^{-1}\mathcal{F}^\bullet$.

On the other hand, in the case when $P = P_1$, one considers the specialization functor for the zero section in a vector bundle. Using the definition of Sp_σ, and hence, of the nearby cycle functor ψ_π in the analytic context ([3] 1.2), one sees that in this case, the functor Sp_σ induces the identity on the category of monodromic complexes.

By extension by zero, let us view objects of $\mathbf{Loc}_F\, M^K(\mathbb{C})$ as sheaves on $M^K(\mathfrak{S})(\mathbb{C})$. From the above, one concludes that the functor Sp_σ induces a functor

$$\mathbf{Loc}_F\, M^K(\mathbb{C}) \longrightarrow \mathbf{Loc}_F\, M^{K_1}(\mathbb{C}) \ ,$$

equally denoted by Sp_σ. For local systems in the image of $\mu_{K,\mathrm{top}}$, we have:

Proposition 2.2 *There is a commutative diagram*

$$\begin{tikzcd} \mathbf{Rep}_F\, P \arrow[r, "\mathrm{Res}^P_{P_1}"] \arrow[d, "\mu_{K,\mathrm{top}}"'] & \mathbf{Rep}_F\, P_1 \arrow[d, "\mu_{K_1,\mathrm{top}}"] \\ \mathbf{Loc}_F\, M^K(\mathbb{C}) \arrow[r, "Sp_\sigma"] & \mathbf{Loc}_F\, M^{K_1}(\mathbb{C}) \end{tikzcd}$$

Proposition 2.3 *There is a commutative diagram*

$$\begin{tikzcd} \mathbf{Rep}_F\, P \arrow[r, "\mathrm{Res}^P_{P_1}"] \arrow[d, "\mu_{K,\mathrm{top}}"'] & \mathbf{Rep}_F\, P_1 \arrow[d, "\mu_{K_1,\mathrm{top}}"] \\ \mathbf{Loc}_F\, M^K(\mathbb{C}) \arrow[d, "j_*"'] & \mathbf{Loc}_F\, M^{K_1}(\mathbb{C}) \arrow[d, "(j_1)_*"] \\ D^b_{\mathrm{c}}\left(M^K(\mathfrak{S})(\mathbb{C}), F\right) \arrow[r, "Sp_\sigma"] & D^b_{\mathrm{c}}\left(M^{K_1}(\mathfrak{S}_{1,[\sigma]})(\mathbb{C}), F\right) \end{tikzcd}$$

Consequently:

Theorem 2.4 *There is a commutative diagram*

$$\begin{tikzcd} D^b\left(\mathbf{Rep}_F\, P\right) \arrow[r, "\mathrm{Res}^P_{P_1}"] \arrow[d, "\mu_{K,\mathrm{top}}"'] & D^b\left(\mathbf{Rep}_F\, P_1\right) \arrow[r, "R(\ )^{\langle\sigma\rangle}"] & D^b\left(\mathbf{Rep}_F\, P_{1,[\sigma]}\right) \arrow[d, "\mu_{\pi_{[\sigma]}(K_1),\mathrm{top}}"] \\ D^b_{\mathrm{c}}\left(M^K(\mathbb{C}), F\right) \arrow[rr, "i^* j_*"] & & D^b_{\mathrm{c}}\left(M^{\pi_{[\sigma]}(K_1)}(\mathbb{C}), F\right) \end{tikzcd}$$

Here, $R(\ )^{\langle\sigma\rangle}$ refers to Hochschild cohomology of the unipotent group $\langle\sigma\rangle \le P_1$.

Proof. By (SP5) and Proposition 2.3, we may assume $P = P_1$. Denote by L_σ the monodromy group of $M^{\pi_{[\sigma]}(K_1)}(\mathbb{C})$ inside $M^{K_1}(\mathfrak{S}_{1,[\sigma]})(\mathbb{C})$. It is generated by the semi-group

$$\Lambda_\sigma(1) := 2\pi i \cdot \Lambda_\sigma \subset U_1(\mathbb{Q})$$

(see the definition before 1.15), and forms a lattice inside $\langle\sigma\rangle$. It is well known that on the image of $\mu_{K,\mathrm{top}}$, the functor $(i_1)^*(j_1)_*$ can be computed via group cohomology of the abstract group L_σ. Since $\langle\sigma\rangle$ is unipotent, its Hochschild cohomology coincides with cohomology of L_σ on algebraic representations.

q.e.d.

Let us reformulate Theorem 2.4 in the language of perverse sheaves ([2]). Since local systems on the space of $\mathbb{C}$-valued points of a smooth complex variety can be viewed as perverse sheaves (up to a shift), we may consider $\mu_{K,\mathrm{top}}$ as exact functor

$$\mathbf{Rep}_F\, P \longrightarrow \mathbf{Perv}_F\, M^K_{\mathbb{C}} \ .$$

By [1] Main Theorem 1.3, the bounded derived category

$$D^b\left(\mathbf{Perv}_F\, M^K_{\mathbb{C}}\right)$$

is canonically isomorphic to $D^b_{\mathrm{c}}(M^K_{\mathbb{C}}, F)$. Theorem 2.4 acquires the following form:

Variant 2.5 *There is a commutative diagram*

$$\begin{tikzcd}
D^b\left(\mathbf{Rep}_F\, P\right) \arrow[r, "\mathrm{Res}^P_{P_1}"] \arrow[d, "\mu_{K,\mathrm{top}}"'] & D^b\left(\mathbf{Rep}_F\, P_1\right) \arrow[r, "R(\ )^{\langle\sigma\rangle}"] & D^b\left(\mathbf{Rep}_F\, P_{1,[\sigma]}\right) \arrow[d, "\mu_{\pi_{[\sigma]}(K_1),\mathrm{top}}"] \\
D^b\left(\mathbf{Perv}_F\, M^K_{\mathbb{C}}\right) \arrow[rr, "i^* j_*[-c]"] & & D^b\left(\mathbf{Perv}_F\, M^{\pi_{[\sigma]}(K_1)}_{\mathbb{C}}\right)
\end{tikzcd}$$

By definition of Shimura data, there is a tensor functor associating to an algebraic F-representation $\mathbb{V}$ of P, for $F \subseteq \mathbb{R}$, a variation of Hodge structure $\mu(\mathbb{V})$ on $\mathfrak{X}$ ([13] 1.18). It descends to a variation $\mu_K(\mathbb{V})$ on $M^K(\mathbb{C})$ with underlying local system $\mu_{K,\mathrm{top}}(\mathbb{V})$. We refer to the functor μ_K as the *canonical construction* of variations of Hodge structure from representations of P.

By [19] Thm. 2.2, the image of μ_K is contained in the category $\mathbf{Var}_F\, M^K_{\mathbb{C}}$ of *admissible* variations, and hence ([16] Thm. 3.27), in the category $\mathbf{MHM}_F\, M^K_{\mathbb{C}}$ of algebraic mixed Hodge modules.

According to [16] 2.30, there is a Hodge theoretic variant of the specialization functor:

$$Sp_\sigma := Sp_{M^{\pi_{[\sigma]}(K_1)}} : \mathbf{MHM}_F\, M^K(\mathfrak{S})_{\mathbb{C}} \longrightarrow \mathbf{MHM}_F\, M^{K_1}(\mathfrak{S}_{1,[\sigma]})_{\mathbb{C}} \ ,$$

which is compatible with Verdier's functor discussed earlier. Since the latter maps local systems on $M^K(\mathbb{C})$ to local systems on $M^{K_1}(\mathbb{C})$ (viewed as sheaves

on the respective compactifications by extension by zero), we see that Sp_σ induces a functor

$$\mathbf{Var}_F\, M_{\mathbb{C}}^K \longrightarrow \mathbf{Var}_F\, M_{\mathbb{C}}^{K_1} \ ,$$

equally denoted by Sp_σ.

Theorem 2.6 *There is a commutative diagram*

$$\begin{tikzcd}
\mathbf{Rep}_F\, P \arrow[r, "\mathrm{Res}^P_{P_1}"] \arrow[d, "\mu_K"'] & \mathbf{Rep}_F\, P_1 \arrow[d, "\mu_{K_1}"] \\
\mathbf{Var}_F\, M_{\mathbb{C}}^K \arrow[r, "Sp_\sigma"] & \mathbf{Var}_F\, M_{\mathbb{C}}^{K_1}
\end{tikzcd}$$

which is compatible with that of 2.2.

Proof. For $\mathbb{V} \in \mathbf{Rep}_F\, P$, denote by $\mathbb{V}_P$ and $\mathbb{V}_{P_1}$ the two variations on the open subset $f^{-1}(M^K(\mathbb{C}))$ of $M^{K_1}(\mathbb{C})$ obtained by restricting $\mu_K(\mathbb{V})$ and $\mu_{K_1}\left(\mathrm{Res}^P_{P_1}(\mathbb{V})\right)$ respectively. By Proposition 2.1, the underlying local systems are identical.

By [13] Prop. 4.12, the Hodge filtrations of $\mathbb{V}_P$ and $\mathbb{V}_{P_1}$ coincide.

Denote the weight filtration on the variation $\mathbb{V}_P$ by $W_\bullet$, and that on $\mathbb{V}_{P_1}$ by $M_\bullet$. Denote by $L_\sigma \subset U_1(\mathbb{Q})$ the monodromy group of $M^{\pi_{[\sigma]}(K_1)}(\mathbb{C})$ inside $M^{K_1}(\mathfrak{S}_{1,[\sigma]})(\mathbb{C})$. Let $T \in L_\sigma$ such that $\frac{1}{2\pi i}T$ or $-\frac{1}{2\pi i}T$ lies in $C(\mathfrak{X}^0, P_1)$. According to Proposition 1.3, the weight filtration of $\log T$ relative to $W_\bullet$ is identical to $M_\bullet$.

Choosing T as the product of the generators of the semi-group

$$\Lambda_\sigma(1) \subset L_\sigma \ ,$$

one concludes that $\mathbb{V}_{P_1}$ carries the limit Hodge structure of $\mathbb{V}_P$ near $M^{\pi_{[\sigma]}(K_1)}$. Using the definition of Sp_σ, and hence, of the nearby cycle functor in the Hodge theoretic context ([16] 2.3), one sees that the two variations $\mu_{K_1} \circ \mathrm{Res}^P_{P_1}\mathbb{V}$ and $Sp_\sigma \circ \mu_K \mathbb{V}$ coincide. **q.e.d.**

Corollary 2.7 *There is a commutative diagram*

$$\begin{tikzcd}
\mathbf{Rep}_F\, P \arrow[r, "\mathrm{Res}^P_{P_1}"] \arrow[d, "\mu_K"'] & \mathbf{Rep}_F\, P_1 \arrow[d, "\mu_{K_1}"] \\
\mathbf{Var}_F\, M_{\mathbb{C}}^K \arrow[d, "j_*"'] & \mathbf{Var}_F\, M_{\mathbb{C}}^{K_1} \arrow[d, "(j_1)_*"] \\
\mathbf{MHM}_F\, M^K(\mathfrak{S})_{\mathbb{C}} \arrow[r, "Sp_\sigma"] & \mathbf{MHM}_F\, M^{K_1}(\mathfrak{S}_{1,[\sigma]})_{\mathbb{C}}
\end{tikzcd}$$

which is compatible with that of 2.3.

Proof. By Theorem 2.6, we have

$$(j_1)^* \, Sp_\sigma \, j_* \circ \mu_K = \mu_{K_1} \circ \mathrm{Res}^P_{P_1} \ .$$

In order to see that the adjoint morphism

$$Sp_\sigma \, j_* \circ \mu_K \longrightarrow (j_1)_* \circ \mu_{K_1} \circ \mathrm{Res}^P_{P_1}$$

is an isomorphism, one may apply the (faithful) forgetful functor to perverse sheaves on $M^{K_1}(\mathfrak{S}_{1,[\sigma]})_{\mathbb{C}}$. There, the claim follows from Proposition 2.3.

q.e.d.

We are ready to prove our main result:

Theorem 2.8 *There is a commutative diagram*

$$\begin{tikzcd}
D^b(\mathbf{Rep}_F\, P) \arrow[r, "\mathrm{Res}^P_{P_1}"] \arrow[d, "\mu_K"'] & D^b(\mathbf{Rep}_F\, P_1) \arrow[r, "R(\ )^{\langle\sigma\rangle}"] & D^b\left(\mathbf{Rep}_F\, P_{1,[\sigma]}\right) \arrow[d, "\mu_{\pi_{[\sigma]}(K_1)}"] \\
D^b\left(\mathbf{MHM}_F\, M^K_{\mathbb{C}}\right) \arrow[rr, "i^* j_*[-c]"] & & D^b\left(\mathbf{MHM}_F\, M^{\pi_{[\sigma]}(K_1)}_{\mathbb{C}}\right)
\end{tikzcd}$$

which is compatible with that of 2.5.

Proof. According to [16] 2.30, we have the equality

$$i^* = i_1^* \circ Sp_\sigma \ .$$

Together with Corollary 2.7, this reduces us to the case $P = P_1$. Now observe that $(i_1)_*$ and $(j_1)_*$ are exact functors on the level of abelian categories $\mathbf{MHM}_F$ ([2] Cor. 4.1.3). $(i_1)^*$ is the left adjoint of $(i_1)_*$ on the level of $D^b(\mathbf{MHM}_F)$. It follows formally that the zeroeth cohomology functor

$$\mathcal{H}^0(i_1)^* : \mathbf{MHM}_F\, M^{K_1}(\mathfrak{S}_{1,[\sigma]})_{\mathbb{C}} \longrightarrow \mathbf{MHM}_F\, M^{\pi_{[\sigma]}(K_1)}_{\mathbb{C}}$$

is right exact, and that $(\mathcal{H}^0(i_1)^*, (i_1)_*)$ constitutes an adjoint pair of functors on the level of $\mathbf{MHM}_F$. In particular, there is an adjunction morphism

$$\mathrm{id} \longrightarrow (i_1)_* \mathcal{H}^0(i_1)^*$$

of functors on $\mathbf{MHM}_F\, M^{K_1}(\mathfrak{S}_{1,[\sigma]})_{\mathbb{C}}$, which induces a morphism of functors on

$$K^b\left(\mathbf{MHM}_F\, M^{K_1}(\mathfrak{S}_{1,[\sigma]})_{\mathbb{C}}\right) \ ,$$

the homotopy category of complexes in $\mathbf{MHM}_F\, M^{K_1}(\mathfrak{S}_{1,[\sigma]})_{\mathbb{C}}$. Denote by q the localization functor from the homotopy to the derived category. We get a morphism in

$$\begin{aligned}
\mathrm{Hom}\,(q, q \circ (i_1)_* \mathcal{H}^0(i_1)^*) &= \mathrm{Hom}\,(q, (i_1)_* \circ q \circ \mathcal{H}^0(i_1)^*) \\
&= \mathrm{Hom}\,((i_1)^* \circ q, q \circ \mathcal{H}^0(i_1)^*) \quad ,
\end{aligned}$$

where Hom refers to morphisms of exact functors. Composition with the exact functor $(j_1)_* \circ \mu_{K_1}$ gives a morphism

$$\eta' \in \operatorname{Hom}\left((i_1)^*(j_1)_* \circ \mu_{K_1} \circ q, q \circ \mathcal{H}^0(i_1)^*(j_1)_* \circ \mu_{K_1}\right) .$$

Assuming the existence of the *total left derived functor*

$$L\left(\mathcal{H}^0(i_1)^*(j_1)_* \circ \mu_{K_1}\right) : D^b\left(\mathbf{Rep}_F P_1\right) \longrightarrow D^b\left(\mathbf{MHM}_F M_{\mathbb{C}}^{\pi_{[\sigma]}(K_1)}\right)$$

for a moment (see (a) below), its universal property ([17] II.2.1.2) says that the above Hom equals

$$\operatorname{Hom}\left((i_1)^*(j_1)_* \circ \mu_{K_1}, L\left(\mathcal{H}^0(i_1)^*(j_1)_* \circ \mu_{K_1}\right)\right) .$$

Denote by

$$\eta : (i_1)^*(j_1)_* \circ \mu_{K_1} \longrightarrow L\left(\mathcal{H}^0(i_1)^*(j_1)_* \circ \mu_{K_1}\right)$$

the morphism corresponding to η'. It remains to establish the following claims:

(a) The functor

$$\mathcal{H}^0(i_1)^*(j_1)_* \circ \mu_{K_1} : \mathbf{Rep}_F P_1 \longrightarrow \mathbf{MHM}_F M_{\mathbb{C}}^{\pi_{[\sigma]}(K_1)}$$

is left derivable.

(b) There is a canonical isomorphism between the total left derived functor

$$L\left(\mathcal{H}^0(i_1)^*(j_1)_* \circ \mu_{K_1}\right)$$

and

$$\mu_{\pi_{[\sigma]}(K_1)} \circ R(\ )^{\langle\sigma\rangle}[c] .$$

(c) η is an isomorphism.

For (a) and (b), observe that up to a twist by c, the variation

$$\mathcal{H}^0(i_1)^*(j_1)_* \circ \mu_K(\mathbb{V})$$

on $M_{\mathbb{C}}^{\pi_{[\sigma]}(K_1)}$ is given by the co-invariants of $\mathbb{V}$ under the local monodromy. This is a general fact about the degeneration of variations along a divisor with normal crossings; see e.g. the discussion preceding [8] (4.4.8). By [11] Thm. 6.10, up to a twist by c (corresponding to the highest exterior power of $\mathrm{Lie}\langle\sigma\rangle$), the co-invariants are identical to $H^c(\langle\sigma\rangle,\ )$.

We are thus reduced to showing that the functor $H^c(\langle\sigma\rangle,\ )$ is left derivable, with total left derived functor $R(\ )^{\langle\sigma\rangle}[c]$. But this follows from standard facts about Lie algebra homology and cohomology (see e.g. [11] Thm. 6.10 and its proof).

(c) can be shown after applying the forgetful functor to perverse sheaves. There, the claim follows from 2.5. **q.e.d.**

Remark 2.9 *If $(P, \mathfrak{X})$ is pure, and $c = \dim\langle\sigma\rangle$ is maximal, i.e., equal to $\dim U_1$, then Theorem 2.8 is equivalent to [8] Thm. (4.4.18). In fact, by 2.8, the recipe to compute $H^q i^* j_* \circ \mu_K(\mathbb{V})$ given on pp. 286/287 of [8] generalizes as follows: The complex*

$$C^\bullet = \Lambda^\bullet(\mathrm{Lie}\langle\sigma\rangle)^* \otimes_F \mathbb{V}$$

carries the diagonal action of P_1 (where the action on $\mathrm{Lie}\langle\sigma\rangle$ is via conjugation). The induced action on the cohomology objects $H^q C^\bullet$ factors through $P_{1,[\sigma]}$ and gives the right Hodge structures via $\mu_{\pi_{[\sigma]}(K_1)}$. In [8], the Hodge and weight filtrations on $C^\bullet$ corresponding to the action of P_1 are made explicit.

Remark 2.10 *Because of 1.14 (b), the isomorphism of Theorem 2.8 does not depend on the cone decomposition $\mathfrak{S}$, which contains $\sigma \times \{p\}$. We leave it to the reader to formulate and prove results like [14] (4.8.5) on the behaviour of the isomorphism of 2.8 under change of the group K, and of the element p.*

Let us conclude the section with a statement on transitivity of degeneration. In addition to the data used so far, fix a face τ of σ. Write

$$i_\tau : M^{\pi_{[\tau]}(K_1)} \hookrightarrow M^K(\mathfrak{S}) \, .$$

$M^{\pi_{[\sigma]}(K_1)}$ lies in the closure of $M^{\pi_{[\tau]}(K_1)}$ inside $M^K(\mathfrak{S})$. Adjunction gives a morphism

$$i^* j_* \circ \mu_K \longrightarrow i^*(i_\tau)_*(i_\tau)^* j_* \circ \mu_K$$

of exact functors from $D^b(\mathbf{Rep}_F\, P)$ to $D^b\left(\mathbf{MHM}_F\, M_{\mathbb{C}}^{\pi_{[\sigma]}(K_1)}\right)$.

Proposition 2.11 *This morphism is an isomorphism.*

Proof. This can be checked on the level of local systems. There, it follows from Theorem 1.11 (i), and standard facts about degenerations along strata in torus embeddings. **q.e.d.**

3 Higher direct images for ℓ-adic sheaves

The main result of this section (Theorem 3.9) provides an ℓ-adic analogue of the formula of 2.8. The main ingredients of the proof are the machinery developed in [14], and our knowledge of the local situation (1.13). 3.6–3.8 are concerned with the problem of extending certain infinite families of étale sheaves to "good" models of a Shimura variety. We conclude by discussing mixedness of the ℓ-adic sheaves obtained via the canonical construction.

With the exception of condition ($\cong$), which will not be needed, we fix the same set of geometric data as in the beginning of Section 2. In particular, the cone σ is assumed smooth, the group K is neat, and $(P, \mathfrak{X})$ satisfies condition (+).

Define $\tilde{M}(\mathfrak{S})$ as the inverse limit of all

$$M^{K'}(\mathfrak{S}) = M^{K'}(P, \mathfrak{X}, \mathfrak{S})$$

for open compact $K' \leq K$. The group K acts on $\tilde{M}(\mathfrak{S})$, and

$$M^K(\mathfrak{S}) = \tilde{M}(\mathfrak{S})/K \ .$$

Inside $\tilde{M}(\mathfrak{S})$ we have the inverse limit $\tilde{M}$ of

$$M^{K'} = M^{K'}(P, \mathfrak{X}) \ , \quad K' \leq K \ ,$$

and the inverse limit $\tilde{M}_{[\sigma]}$ of all

$$M^{K'_{1,[\sigma]}} = M^{K'_{1,[\sigma]}}(P_{1,[\sigma]}, \mathfrak{X}_{1,[\sigma]})$$

for open compact $K'_{1,[\sigma]} \leq K_{1,[\sigma]} := \pi_{[\sigma]}(K_1)$. We get a commutative diagram

$$\begin{tikzcd}
\tilde{M} \arrow[r, hook, "\tilde{j}"] \arrow[d] & \tilde{M}(\mathfrak{S}) \arrow[d] & \tilde{M}_{[\sigma]} \arrow[l, hook', "\tilde{i}"'] \arrow[d] \\
\tilde{M}/K_1 \arrow[r, hook, "j'"] \arrow[d, "\varphi"'] & \tilde{M}(\mathfrak{S})/K_1 \arrow[d, "\tilde{\varphi}"'] & \tilde{M}_{[\sigma]}/K_{1,[\sigma]} \arrow[l, hook', "i'"'] \arrow[d, equal] \\
M^K = \tilde{M}/K \arrow[r, hook, "j"] & M^K(\mathfrak{S}) = \tilde{M}(\mathfrak{S})/K & M^{\pi_{[\sigma]}(K_1)} \arrow[l, hook', "i"']
\end{tikzcd} \tag{3.1}$$

Proposition 3.2 *The morphism*

$$\tilde{\varphi} : \tilde{M}(\mathfrak{S})/K_1 \longrightarrow M^K(\mathfrak{S}) = \tilde{M}(\mathfrak{S})/K$$

is étale near the stratum

$$M^{\pi_{[\sigma]}(K_1)} = \tilde{M}_{[\sigma]}/K_{1,[\sigma]} \ .$$

Proof. By Theorem 1.13, the map $\tilde{\varphi}$ induces an isomorphism of the respective formal completions along our stratum. The claim thus follows from [7] Prop. (17.6.3). **q.e.d.**

Let $\mathrm{Tor\,Mod}_K$ be the category of all continuous discrete torsion K-modules. The left vertical arrow of (3.1) gives an evident functor

$$\mu_K : \mathrm{Tor\,Mod}_K \longrightarrow \mathbf{Et}\, M^K$$

into the category of étale sheaves on M^K; since K is neat, this functor is actually an exact tensor functor with values in the category of lisse sheaves.

Similar remarks apply to K_1 or $\pi_{[\sigma]}(K_1)$ in place of K. We are interested in the behaviour of the functor

$$i^* j_* : D^+(\mathbf{Et}\, M^K) \longrightarrow D^+(\mathbf{Et}\, M^{\pi_{[\sigma]}(K_1)})$$

on the image of μ_K. From 3.2, we conclude:

Proposition 3.3 *(i) The two functors*

$$i^* j_* \quad , \quad (i')^* j'_* \circ \varphi^* : D^+(\mathbf{Et}\, M^K) \longrightarrow D^+(\mathbf{Et}\, M^{\pi_{[\sigma]}(K_1)})$$

are canonically isomorphic.
(ii) The two functors

$$i^* j_* \circ \mu_K \ , \ (i')^* j'_* \circ \mu_{K_1} \circ \mathrm{Res}^K_{K_1} : D^+(\mathrm{Tor\,Mod}_K) \longrightarrow D^+(\mathbf{Et}\, M^{\pi_{[\sigma]}(K_1)})$$

are canonically isomorphic. Here, $\mathrm{Res}^K_{K_1}$ *denotes the pullback via the monomorphism*

$$K_1 \longrightarrow K \ , \ k_1 \longmapsto p^{-1} \cdot k_1 \cdot p \ .$$

Proof. (i) is smooth base change, and (ii) follows from (i). **q.e.d.**

Write K_σ for $\ker(\pi_{[\sigma]}\,|_{K_1}) = K_1 \cap \langle \sigma \rangle(\mathbb{A}_f)$.

Theorem 3.4 *There is a commutative diagram*

$$\begin{tikzcd}
D^+\,(\mathrm{Tor\,Mod}_K) \arrow[r, "\mathrm{Res}^K_{K_1}"] \arrow[d, "\mu_K"'] & D^+\,(\mathrm{Tor\,Mod}_{K_1}) \arrow[r, "R(\ )^{K_\sigma}"] & D^+(\mathrm{Tor\,Mod}_{\pi_{[\sigma]}(K_1)}) \arrow[d, "\mu_{\pi_{[\sigma]}(K_1)}"] \\
D^+\,(\mathbf{Et}\, M^K) \arrow[rr, "i^* j_*"] & & D^+\,(\mathbf{Et}\, M^{\pi_{[\sigma]}(K_1)})
\end{tikzcd}$$

Here, $R(\ )^{K_\sigma}$ *refers to continuous group cohomology of* K_σ*.*

Proof. We need to show that the diagram

$$\begin{tikzcd}
D^+\,(\mathrm{Tor\,Mod}_{K_1}) \arrow[r, "R(\ )^{K_\sigma}"] \arrow[d, "\mu_{K_1}"'] & D^+(\mathrm{Tor\,Mod}_{\pi_{[\sigma]}(K_1)}) \arrow[d, "\mu_{\pi_{[\sigma]}(K_1)}"] \\
D^+(\mathbf{Et}\, \tilde{M}/K_1) \arrow[r, "(i')^* j'_*"] & D^+\,(\mathbf{Et}\, M^{\pi_{[\sigma]}(K_1)})
\end{tikzcd}$$

commutes. The proof of this statement makes use of the full machinery developed in the first two section of [14].

In fact, [14] Prop. (4.4.3) contains the analogous statement for the (coarser) stratification of $M^K(\mathfrak{S})$ induced from the canonical stratification of the *Baily–Borel compactification* of M^K. One faithfully imitates the proof, observing that [14] (1.9.1) can be applied because the upper half of (3.1) is cartesian up to nilpotent elements. The statement on ramification along a stratum in [14] (3.11) holds for arbitrary, not just pure Shimura data.

q.e.d.

Remark 3.5 *Because of Remark 1.14 (b), the isomorphism of 3.4 does not depend on the cone decomposition $\mathfrak{S}$ containing $\sigma \times \{p\}$.*

Fix a set $\mathcal{T} \subset \operatorname{Tor}\operatorname{Mod}_K$, let $E = E(P, \mathfrak{X})$ be the field of definition of our varieties, and write O_E for its ring of integers. Consider a *model*

$$\mathcal{M}^K \overset{j}{\hookrightarrow} \mathcal{M}^K(\mathfrak{S}) \overset{i}{\hookleftarrow} \mathcal{M}^{\pi_{[\sigma]}(K_1)}$$

of

$$M^K \overset{j}{\hookrightarrow} M^K(\mathfrak{S}) \overset{i}{\hookleftarrow} M^{\pi_{[\sigma]}(K_1)}$$

over O_E, i.e., normal schemes of finite type over O_E, an open immersion j and an immersion i whose generic fibres give the old situation over E; we require also that the generic fibres lie dense in their models. (Finitely many special fibres of our models might be empty.)

Assume

(1) All sheaves in $\mu_K(\mathcal{T})$ extend to lisse sheaves on $\mathcal{M}^K$.

(2) For any $S \in \mu_K(\mathcal{T})$ and any $q \geq 0$, the extended sheaf $\mathcal{S}$ on $\mathcal{M}^K$ satisfies the following:

$$i^* R^q j_* \mathcal{S} \in \mathbf{Et}\, \mathcal{M}^{\pi_{[\sigma]}(K_1)} \text{ is lisse.}$$

Then the generic fibre of $i^* R^q j_* \mathcal{S}$ is necessarily equal to $i^* R^q j_* S$, i.e., it is given by the formula of 3.4. So $i^* R^q j_* \mathcal{S}$ is the unique lisse extension of $i^* R^q j_* S$ to $\mathcal{M}^{\pi_{[\sigma]}(K_1)}$. Observe that if $\mathcal{T}$ is finite, then conditions (1) and (2) hold after passing to an open sub-model of any given model.
If $\mathcal{T}$ is an abelian subcategory of $\operatorname{Tor}\operatorname{Mod}_K$ and (1) holds, then (2) needs to be checked only for the simple noetherian objects in $\mathcal{T}$.

Let us show how to obtain a model as above for a *particular* choice of $\mathcal{T}$: Fix a prime ℓ, write

$$\mathrm{pr}_\ell : P(\mathbb{A}_f) \longrightarrow P(\mathbb{Q}_\ell)$$

and $K_\ell := \mathrm{pr}_\ell(K)$. Denote by $\mathcal{T}_\ell \subset \operatorname{Tor}\operatorname{Mod}_{K_\ell} \subset \operatorname{Tor}\operatorname{Mod}_K$ the abelian subcategory of $\mathbb{Z}_\ell$-torsion K_ℓ-modules. The quotient K_ℓ of K corresponds to a certain part of the "Shimura tower"

$$(M^{K'})_{K'} \, ,$$

namely the one indexed by the open compact $K' \leq K$ containing the kernel of $\mathrm{pr}_\ell\,|_K$. According to [14] (4.9.1), the following is known:

Proposition 3.6 *There exists a model $\mathcal{M}^K$ such that all the sheaves in*

$$\mu_K(\mathcal{T}_\ell)$$

extend to lisse sheaves on $\mathcal{M}^K$. Equivalently, the whole étale K_ℓ-covering of M^K considered above extends to an étale K_ℓ-covering of $\mathcal{M}^K$.

Proof. Write L for the product of ℓ and the primes dividing the order of the torsion elements in K_ℓ; thus K_ℓ is a pro-L-group. Let S be a finite set in $\mathbf{Spec}\, O_E$ containing the prime factors of L, and $\mathcal{M}^K$ a model of M^K over O_S which is the complement of an NC-divisor relative to O_S in a smooth, proper O_S-scheme.

We give a construction of a suitable enlargement S' of S such that the claim holds for the restriction of $\mathcal{M}^K$ to $O_{S'}$.

First, assume that P is a torus. Recall ([13] 2.6) that the Shimura varieties associated to tori are finite over their reflex field. Since Shimura varieties are normal, each M^K is the spectrum of a finite product E_K of number fields. But then the K_ℓ-covering corresponds to an *abelian* K_ℓ-extension

$$\tilde{E}/E_K \ .$$

By looking at the kernel of the reduction map to $\mathrm{GL}_N(\mathbb{Z}/\ell^f\mathbb{Z})$, $\ell^f \geq 3$, one sees that there is an intermediate extension

$$\tilde{E}/F/E_K$$

finite over E_K, such that $\tilde{E}/F$ is a $\mathbb{Z}_\ell^r$-extension. Hence the only primes that ramify in $\tilde{E}/F$ are those over ℓ, and one adds to S the finitely many primes which ramify in F/E_K.

In the general case, choose an embedding

$$e : (T, \mathcal{Y}) \longrightarrow (P, \mathfrak{X})$$

of Shimura data, with a torus T such that $E = E(P, \mathfrak{X})$ is contained in $E(T, \mathcal{Y})$ ([13] Lemma 11.6), and finitely many $K_m^T \leq T(\mathbb{A}_f)$ and $p_m \in P(\mathbb{A}_f)$ such that the maps

$$[\cdot p_m] \circ [e] : M^{K_m^T}(T, \mathcal{Y}) \longrightarrow M^K(P, \mathfrak{X})$$

are defined and meet all components of M^K ([13] Lemma 11.7). Each $M^{K_m^T}$ equals the spectrum of a product F_m of number fields.

Define $x_m \in M^K(F_m)$ as the image of $[\cdot p_m] \circ [e]$. Let $S_m \subset \mathbf{Spec}\, O_{F_m}$ denote the set of bad primes for $M^{K_m^T}$ and $(K_m^T)_\ell$, plus a suitable finite set such that x_m extends to a section of $\mathcal{M}^K$ over O_{S_m}.

Enlarge $S = S((T, \mathcal{Y}), e, p_m)$ so as to contain all primes which ramify in some F_m, and those below a prime in some S_m. We continue to write S for the enlargement, and $\mathcal{M}^K$ and x_m for the objects obtained via restriction to O_S.

We claim that with these choices, the whole étale K_ℓ-covering of M^K extends to an étale K_ℓ-covering of $\mathcal{M}^K$.

Let M^0 and $\mathcal{M}^0$ be connected components of M^K and $\mathcal{M}^K$. We have to

show that the map

$$s : \pi_1(M^0) \longrightarrow K_\ell$$

given by the K_ℓ-covering factors through the epimorphism

$$\beta : \pi_1(M^0) \longrightarrow\!\!\!\!\rightarrow \pi_1(\mathcal{M}^0) \; .$$

There is an m and intermediate field extensions

$$F_m/F'/F/E$$

such that M^0 is a scheme over F with geometrically connected fibres, and such that x_m induces an F'-valued point of M^0. Since $\mathcal{M}^0$ is normal, $\mathcal{M}^0$ is a scheme over the integral closure O_{S_F} of O_S. By [15] 4.2–4.4, there is a commutative diagram of exact sequences

$$\begin{tikzcd}
1 \arrow[r] & \pi_1(\overline{M^0}) \arrow[r] \arrow[d, "\alpha"'] & \pi_1(M^0) \arrow[r] \arrow[d, "\beta'"'] & \mathrm{Gal}(\overline{F}/F) \arrow[r] \arrow[d, "\gamma"] & 1 \\
1 \arrow[r] & \pi_1^L(\overline{M^0}) \arrow[r] & \pi_1'(\mathcal{M}^0) \arrow[r] & \pi_1(\mathbf{Spec}\, O_{S_F}) \arrow[r] & 1
\end{tikzcd}$$

Here, $\pi_1^L(\overline{M^0})$ is the largest pro-L-quotient of $\pi_1(\overline{M^0})$, the fundamental group of $\overline{M^0} := M^0 \otimes_F \overline{F}$, and $\pi_1'(\mathcal{M}^0)$ is a suitable quotient of $\pi_1(\mathcal{M}^0)$. Hence all vertical arrows are surjections.

Clearly $\ker \alpha$ is contained in $\ker s$; we thus get a map

$$s' : \pi_1(M^0)/\ker \alpha \longrightarrow K_\ell \; .$$

We have to check that

$$\ker \gamma = \ker \beta' / \ker \alpha \subset \pi_1(M^0)/\ker \alpha$$

is contained in $\ker s'$. But $\ker \gamma$ remains unchanged under passing to the extension F'/F, which is unramified outside S_F. There, the corresponding exact sequence splits thanks to the existence of x_m.
The map

$$\ker \gamma \longrightarrow \pi_1(M^0)$$

is induced by pullback via $[\cdot p_m] \circ [e]$, and by construction its image is contained in $\ker s$. **q.e.d.**

This takes care of condition (1).

Lemma 3.7 *Up to isomorphism, there are only finitely many simple objects in $\mathcal{T}_\ell$.*

Proof. There is a normal subgroup $K'_\ell \leq K_\ell$ of finite index which is a projective limit of ℓ-groups. Write $\mathcal{T}'_\ell$ for the subcategory of $\mathrm{Tor\,Mod}_{K'_\ell}$ of $\mathbb{Z}_\ell$-torsion modules. Since any element of order ℓ^n in $\mathrm{GL}_r(\mathbb{F}_\ell)$ is unipotent, any simple non-trivial object in $\mathcal{T}'_\ell$ is isomorphic to the trivial representation $\mathbb{Z}/\ell\mathbb{Z}$ of K'_ℓ.

Therefore, the simple objects in $\mathcal{T}_\ell$ all occur in the Jordan–Hölder decomposition of

$$\operatorname{Ind}_{K'_\ell}^{K_\ell} \operatorname{Res}_{K'_\ell}^{K_\ell}(\mathbb{Z}/\ell\mathbb{Z}) \ .$$

q.e.d.

Proposition 3.8 *Conditions (1) and (2) hold for a suitable open submodel of any model as in 3.6.*

Proof. By generic base change ([4] Thm. 1.9), condition (2) can be achieved for any single constructible sheaf $\mathcal{S}$ on $\mathcal{M}^K$, which is lisse on the generic fibre. The claim follows from 3.7 by applying the long exact sequences associated to $i^* Rj_*$. **q.e.d.**

Fix a finite extension $F = F_\lambda$ of $\mathbb{Q}_\ell$. By passing to projective limits, we get an exact tensor functor

$$\mu_K : \mathbf{Rep}_F \, P \longrightarrow \mathbf{Et}^l_F \, M^K$$

into the category of lisse λ-adic sheaves on M^K ([14] (5.1)). We refer to μ_K as the *canonical construction* of λ-adic sheaves from representations of P. Denote by $D^b_{\mathrm{c}}(?, F)$ Ekedahl's bounded "derived" category of constructible F-sheaves ([6] Thm. 6.3). Consider the functor

$$i_* j^* : D^b_{\mathrm{c}}(M^K, F) \longrightarrow D^b_{\mathrm{c}}(M^{\pi_{[\sigma]}(K_1)}, F) \ .$$

From Theorem 3.4, we obtain the main result of this section:

Theorem 3.9 *There is a commutative diagram*

$$\begin{tikzcd}
D^b\left(\mathbf{Rep}_F \, P\right) \arrow[r, "\operatorname{Res}^P_{P_1}"] \arrow[d, "\mu_K"'] & D^b\left(\mathbf{Rep}_F \, P_1\right) \arrow[r, "R(\ )^{\langle\sigma\rangle}"] & D^b\left(\mathbf{Rep}_F \, P_{1,[\sigma]}\right) \arrow[d, "\mu_{\pi_{[\sigma]}(K_1)}"] \\
D^b_{\mathrm{c}}\left(M^K, F\right) \arrow[rr, "i^* j_*"] & & D^b_{\mathrm{c}}\left(M^{\pi_{[\sigma]}(K_1)}, F\right)
\end{tikzcd}$$

Here, $\operatorname{Res}^P_{P_1}$ *denotes the pullback via the monomorphism*

$$P_{1,F} \longrightarrow P_F \ , \ p_1 \longmapsto \pi_\ell(p)^{-1} \cdot p_1 \cdot \pi_\ell(p) \ ,$$

and $R(\ )^{\langle\sigma\rangle}$ *is Hochschild cohomology of the unipotent group* $\langle\sigma\rangle$.

Proof. Since $\langle\sigma\rangle$ is unipotent, $R(\ )^{K_\sigma}$ and $R(\ )^{\langle\sigma\rangle}$ agree. **q.e.d.**

Let us note a refinement of the above. Consider smooth models

$$\mathcal{M}^K \overset{j}{\hookrightarrow} \mathcal{M}^K(\mathfrak{S}) \overset{i}{\hookleftarrow} \mathcal{M}^{\pi_{[\sigma]}(K_1)}$$

satisfying conditions (1), (2) for $\mathcal{T}_\ell$. Thus all the sheaves in the image of μ_K extend to $\mathcal{M}^K$; in particular they can be considered as (locally constant) *perverse F-sheaves* in the sense of [9]:

$$\mu_K : \mathbf{Rep}_F \, P \longrightarrow \mathbf{Perv}_F \, \mathcal{M}^K \subset D^b_{\mathrm{c}}(\mathfrak{U}\mathcal{M}^K, F)$$

(notation as in [9]). Consider the functor

$$i_* j^* : D^b_{\mathrm{c}}(\mathfrak{U}\mathcal{M}^K, F) \longrightarrow D^b_{\mathrm{c}}(\mathfrak{U}\mathcal{M}^{\pi_{[\sigma]}(K_1)}, F) \ .$$

Variant 3.10 *There is a commutative diagram*

$$\begin{tikzcd} D^b(\mathbf{Rep}_F P) \arrow[r, "\mathrm{Res}^P_{P_1}"] \arrow[d, "\mu_K"'] & D^b(\mathbf{Rep}_F P_1) \arrow[r, "R(\ )^{\langle\sigma\rangle}"] & D^b\left(\mathbf{Rep}_F P_{1,[\sigma]}\right) \arrow[d, "\mu_{\pi_{[\sigma]}(K_1)}"] \\ D^b_{\mathrm{c}}\left(\mathfrak{U}\mathcal{M}^K, F\right) \arrow[rr, "i^* j_*[-c]"] & & D^b_{\mathrm{c}}\left(\mathfrak{U}\mathcal{M}^{\pi_{[\sigma]}(K_1)}, F\right) \end{tikzcd}$$

Remark 3.11 *As in 3.4, the isomorphism*

$$\mu_{\pi_{[\sigma]}(K_1)} \circ R(\ )^{\langle\sigma\rangle} \circ \mathrm{Res}^P_{P_1} \xrightarrow{\sim} i^* j_* \circ \mu_K[-c]$$

does not depend on the cone decomposition $\mathfrak{S}$ containing $\sigma \times \{p\}$. It is possible, as in [14] (4.8.5), to identify the effect on the isomorphism of change of the group K and of the element p. Similarly, one has an ℓ-adic analogue of Proposition 2.11.

In the above situation, consider the *horizontal stratifications* ([9] page 110) $\mathbf{S} = \{\mathcal{M}^K\}$ of $\mathcal{M}^K$ and $\mathbf{T} = \{\mathcal{M}^{\pi_{[\sigma]}(K_1)}\}$ of $\mathcal{M}^{\pi_{[\sigma]}(K_1)}$. Write $L_{\mathbf{S}}$ and $L_{\mathbf{T}}$ for the sets of extensions to the models of irreducible objects of $\mu_K(\mathbf{Rep}_F P)$ and $\mu_{\pi_{[\sigma]}(K_1)}(\mathbf{Rep}_F P_{1,[\sigma]})$ respectively. In the terminology of [9] Def. 2.8, we have the following:

Proposition 3.12 *$i_* j^*$ is $(\mathbf{S}, L_{\mathbf{S}})$-to-$(\mathbf{T}, L_{\mathbf{T}})$-admissible.*

Proof. This is [9] Lemma 2.9, together with Theorem 3.9. **q.e.d.**

It is conjectured ([12] § 6; [14] (5.4.1); [19] 4.2) that the image of μ_K consists of *mixed sheaves with a weight filtration*; furthermore, the filtration should be the one induced from the weight filtration of representations of P. Let us refer to this as the *mixedness conjecture* for $(P, \mathfrak{X})$; cmp. [14] (5.5)–(5.6) and [19] pp 112–116 for a discussion. The conjecture is known if every $\mathbb{Q}$-simple factor of G^{ad} is *of abelian type* ([14] Prop. (5.6.2), [19] Thm. 4.6 (a)).

Proposition 3.13 *If the mixedness conjecture holds for $(P, \mathfrak{X})$, then it holds for any rational boundary component $(P_1, \mathfrak{X}_1)$.*

Proof. By [19] Thm. 4.6, it suffices to check that $\mu_{\pi_{[\sigma]}(K_1)}(\mathbb{W})$ is mixed for some faithful representation $\mathbb{W}$ of $P_{1,[\sigma]}$. By [10] Thm. 11.2, there is a representation $\mathbb{V}$ of P and a one-dimensional subspace $\mathbb{V}' \subset \mathbb{V}$ such that

$$\langle\sigma\rangle = \mathrm{Stab}_P(\mathbb{V}') \ .$$

Since $\langle\sigma\rangle$ is unipotent, we have

$$\mathbb{V}' \subset \mathbb{W} := H^0(\langle\sigma\rangle, \mathbb{V}) .$$

$\mathbb{W}$ is a faithful representation of $P_{1,[\sigma]}$, and by Theorem 3.9, $\mu_{\pi_{[\sigma]}(K_1)}(\mathbb{W})$ is a cohomology object of the complex

$$i^* j_* \circ \mu_K(\mathbb{V}) .$$

It is therefore mixed ([5] Cor. 6.1.11). **q.e.d.**

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